Chapter 7

Basic Principles of Physics

7.1 Symmetry

Symmetry is one of those concepts that occur in our everyday language and also in physics. There is some similarity in the two usages, since, as is usually the case, the physics usage generally grew out of theeveryday usage but is more precise. Let's start with the general usage. Synonyms for symmetry are words like balanced or well formed. We most often use the idea in terms of a work of art. The following 4th century greek statue, Figure 7.1 on page 184, of a praying boy is a beautiful work of art. This is attributable to the form and balance. The figure has an almost exact bilateral, axial reflection, symmetry. A bilateral symmetry is a well defined mathematical operation on the figure: Establish a mean central axis and place a mirror there to reflect every point on the object in the plane plane of the mirror. You recover almost the same figure. In fact a Platonist would attribute the beauty in the piece to the presence of the mathematical symmetry. Of course, for this case, the symmetry is not exact but approximate.

These ideas about symmetry can be generalized and at the same time made more specific. In art and in physics, the idea is that you perform some algorithmic or well specified operation to the figure or system of interest. If you recover the same figure or system then you have a symmetry. Later on we will get very specific as to the definition of symmetry but the basic idea that you see here will endure. There is some change that you can make and if after you make the change you have basically the same thing that you started with, you say that you have a symmetry. If you recover almost the same figure or system, you have what is called a slightly broken symmetry or approximate symmetry.

Figure 7.1: **Praying Boy** In art, as it will turn out to be the case in physics, there is a sense of beauty associated with balanced or symmetric figures. This ancient greek statue of a praying boy has an approximate bilateral symmetry.

The first issue is to understand the idea of making a change. In order to differentiate the parts of this problem, we will call these changes transformations. There are obviously many transformations that you can perform both in physics and in art. Moving the figure to the side is an especially simple example. The set of operations that are shifting of the figure Is an example of what is called a translation. In art, if the figure is the same after it has been translated, the figure possess translation symmetry; the transformation is a translation and there is a symmetry if the figure is the identical to the original. In most cases in art with translation symmetry, the amount of translation that reproduces the original image is an integer multiple of some fixed amount, see Figure 7.3 on page 185. This is an example of a discrete translation symmetry. Our earlier example of bilateral transformations or mirror images is also an example of a discrete family of transformations. This is an especially simple family since, if you do the transformation twice, you have not done anything. There are thus only two transformations in the bilateral set: mirror image or leave alone. The case of Figure 7.3 on page 185, there are many translations that produce a symmetry. In fact,

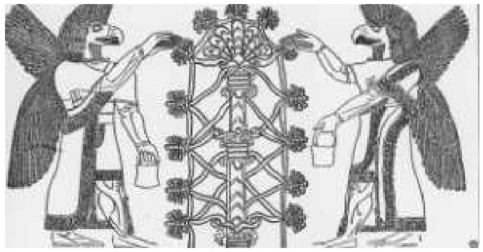

Figure 7.2: **Ancient Drawing** This ancient drawing shows an example of bilateral or reflection symmetry. Close inspection reveals that the symmetry is broken in an interesting way.

there is an infinite countable set of transformations, i. e. the transformations that produce a symmetry can be mapped onto the set of integers. Note that any combination of translations in the set of discrete translations is also a discrete translation. This is an important property of a family of transformations: they always contain in the family all combinations of the elements. In addition, they also contain the element that is no change and they also always contain an element that undoes what another element does. In the bilateral case, the only non-trivial element undoes itself if it is applied again. For the case of Figure 7.3 on page 185, you can reverse the direction of the original translation and shift the same amount.

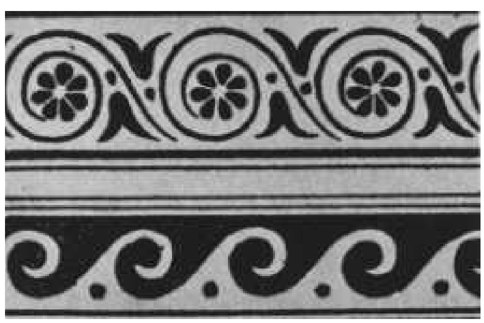

Figure 7.3: **Borders** Note how border images tend to have discrete translation symmetry. It also has bilateral symmetry. Of course, we are assuming that the border extends indefinitely in both directions.

Another well known example of transformations in art and physics is rotations about an axis. Snowflakes are an interesting example, see Figure 7.4 on page 186. They possess a discrete rotational symmetry. Rotations of an integer multiple of $\frac{2\pi}{6}$, reproduce the original image. Again, like the bilateral transformation, after so many of these rotations you can get back to doing nothing. This is a more interesting example of the discrete transformations with a finite number of elements than the bilateral case besides doing nothing there are five non-trivial rotations, $\frac{\pi}{3}$, $\frac{2\pi}{3}$, π, $\frac{4\pi}{3}$, and $\frac{5\pi}{3}$. In addition, the snowflake also has a bilateral. In fact since it has the discrete rotations, it actually has several bilateral transformations. These are along axis at 0, $\frac{2\pi}{12}$, $\frac{4\pi}{12}$, $\frac{6\pi}{12}$, $\frac{8\pi}{12}$, and $\frac{10\pi}{12}$. These being combinations of the bilaterals and rotations.

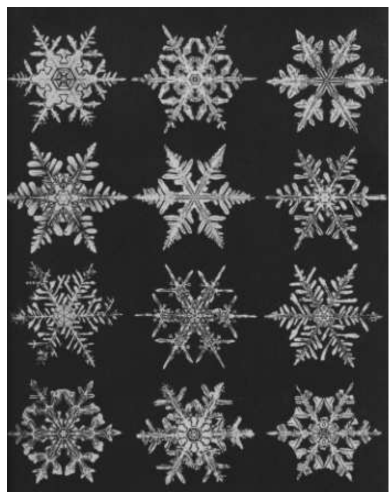

Figure 7.4: **Snowflakes** Snowflakes provide an excellent natural example of a system with a discrete rotation symmetry. It also has a bilateral symmetry and, since it has a rotational symmetry, actually has several bilaterals.

As stated earlier, symmetry is a change to a system or, in the case of art, a figure or a statue that is not an important change. From these examples it is important to realize that to have a symmetry, you need a set of changes to the figure **and** then a criteria for these not being an important change. In the case of art, the criteria for not being important is that the pieces fall on top of each other. You could have a much more relaxed definition of unimportant change. For example consider the world of three sided figures whose sides are straight lines, triangles. If your criteria for unimportant change is that after the transformation you still have a triangle, then any transformation short of opening or bending one of the sides will be a symmetry. You could have a more restrictive criteria such as that the triangles be similar. In this case, rotations and rescaling all lengths would be a symmetry but changing the size of one of the sides and not the others would not. It is important to keep in mind that the concept of symmetry is a two step process – a family of transformations and a rule about what is an important change.

Although we did not discuss it in those terms, we have already had an example of a symmetry in physics when we looked at the change in scale when we discussed dimensional analysis, see Section 2.5 on page 44. If we change the scale of length, all the numbers change but the things that happen still happen; it doesn't matter whether you make the measurements in the cgs system, the mks system, or english system, the physics is the same. We can use this as a rather loose definition of what we mean by a symmetry in physics. As we develop our vocabulary morefully, we can make this definition much more precise.

For all of the discussion so far we have defined the transformations as changes to the figure; rotate the figure by $\frac{2\pi}{6}$. With the example of change in scale, we can see a different but clearly equivalent approach. Instead of stretching the figure, we can just use a smaller length scale to discuss its size. In the old perspective, you can also look at it as if all lengths increased and the unit of length stayed the same. Here you now say that the figure stays the same and the unit of length changes. This is the difference between the active and the passive view of a transformation. In the active view, you change the figure, in the passive view the figure is left unchanged but the observers perspective is changed. In the active view, you then have another perspective

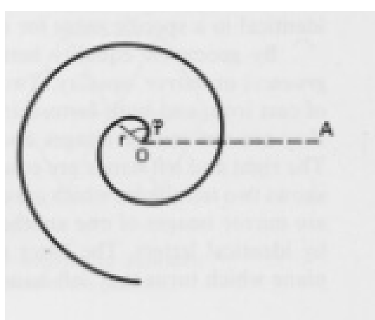

Figure 7.5: **Spiral** The spiral is generated by stretching the radius as you rotate. This is an example of a situation in which you combine two simple transformations to generate a figure with symmetry.

in symmetry. You can use the transformation to generate a figure that will automatically be symmetric. An extreme example of a symmetry is the infinitely long straight line. It satisfies bilateral symmetry about every point. It satisfies a translation symmetry of any amount. It is homogeneous, same everywhere, and isotropic, same in both directions which are all the directions that it has. In turn, you can think of the stiaght line as the figure that is generated by translating a point to generate a continuous figure. Another important example is the circle. As a figure it is symmetric under rotations about the center. It can also be considered the locus of points that are equidistant from some fixed point and is generated by rotation of point at the appropriate distance from the center.

As in the snowflake example, Figure 7.4 on page 186, the family of transformations used in an active transformation includes all possible combinations of all of the elements of the family. In many cases, the resulting transformations can be a little surprising. The spiral is a shape that is generated by a compound of several simpler operations, stretch the radius as you rotate. In this case, the figure has a symmetry if as you translate in angle you stretch the distance from the origin. An interesting related example taken from biology is the shell seen in Figure 7.6 on page 188.

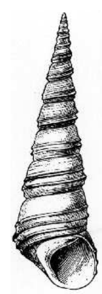

Figure 7.6: **Shell** The shell is an interesting example of a symmetric system. As you rotate, you translate and stretch the radius.

7.2 The Nature of Symmetry in Physics

In many respects, symmetry in physics is very similar to that in art; there are families of transformations that lead to unimportant changes in the situation. The differences deal with the things on which the transformations act and the definition of unimportant. As expected, in addition, the language that described the actions are more precise and abstract. We will also categorize the transformations of physics in a formal way and use these labels to describe important results.

7.2.1 Discrete Transformations

These are changes that can only be applied in discrete steps. Bilateral or mirror symmetry about a plane is an example from art. For the snow flakes or a particular snow flake, see Figure 7.4 on page 186, the rotations at $\theta = n\frac{2\pi}{6}$ for $n = 0, 1, 2, \cdots, 5$ is an example of a family of discrete rotations that produce a symmetry for a snowflake. The rotated snow flake falls onto the image of that snow flakes original configuration exactly.

The example in physics that corresponds to bilateral symmetry is called a spatial inversion which is to replace places in one directions by their opposite. In a world with on space dimension, replace x by $-x$. In a world with three spatial directions, replace (x, y, z) with $(-x, y, z)$. This is like placing a mirror in the plane $y = 0, z = 0$. This is obviously a discrete transformation. You also note that, if it is applied twice, there is no change. It is said to be a discrete transformation of cycle two; it has two elements, do nothing, the identity transformation, and the inversion. There are many discrete transformations of cycle two: if you have identical particles, you can interchange the particles, you can invert the time, you can do a spatial inversion along the y or z axis, ...

There are, of course, discrete transformations with cycles higher than two. The snowflake example from art carries over to physics. Rotations about the origin by an angle of $\frac{2\pi}{n}$ is an example of a discrete transformation with n cycles.

You can also have a family of discrete transformations that have an infinite number of elements. In one spatial dimension, you can shift the origin by a fixed amount a, see Figure 7.3 on page 185. You can do this any number of times generating a set of transformations that has a countable infinite number of members.

It is important to realize that the method by which the members of a family of discrete transformations are labeled must itself be a discrete set of labels and that the members of a discrete set of transformations cannot be labeled by a continuous variable.

7.2.2 Continuous Transformations

Continuous transformations are changes that can be applied for arbitrarily small changes. The labeling of the transformations is a continuous parameter. Rotations about a point are a valuable example. In art, a world of concentric rings would enjoy a symmetry for rotations about the center point. These changes in angle can take any value from zero to 2π. This idea is carried over to physics. In a three dimensional space, rotations about an axis are a family of transformations. These transformations are an example of continuous transformations. Other obvious examples are translations in space and time. Changes in the scale of length discussed in Sections 1.5.1 on page 25, and 2.5.2 on page 47 is also a continuous set of transformations. **Again it is important to realize that a continuous family of transformations can only be labeled by a continuous variable.**

It is possible to make a discrete family of transformations from subsets of continuous transformations such as the set of rotations used in the snowflake example of Figure 7.4 on page 186. Of course, the reverse process is not possible; you cannot make a continuous family of transformations from a subset of a discrete family no matter how large the set of discrete transformations.

7.2.3 Identity Transformation

Another principle that is essential to the mathematical consistency of transformation theory is that a complete set of transformations must include an identity transformation. That is one that changes nothing. For any collection of transformations, The example $n = 0$ in the discrete case above is an identity transformation. Note that $n = 6m$ where $m = 1, 2, 3...$ is also the identity and we already had it in the set of transformations. In fact, any transformation in which $n > 6$ is the same as the transformation $n' = \mathrm{mod}_6(n)$.

7.2.4 Examples of symmetry in situations like physics

You are planning a trip between Austin and College Station. There are several routes.

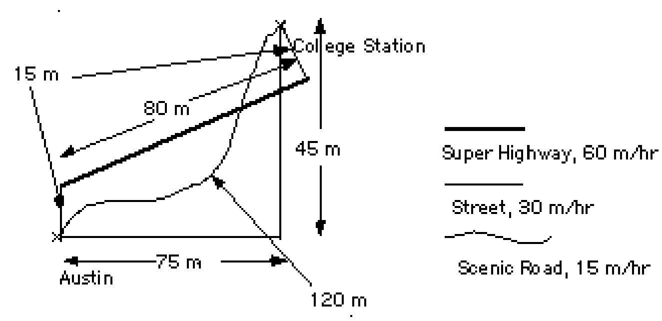

Figure 7.7: **Paths to Texas A&M** Miles to AM

There are several criteria that you can use to select the route: least time, least distance, see most trees and hills - one hill is worth a dozen trees. There are several changes that you can make in the system: interchange Austin and College Station, interchange super highways and streets, make the speed limit 50 $\frac{m}{hr}$, measure all distances in feet. These are all discrete transformations. You could shift the entire thing a distance x to the east and we all know that as you go east there are no longer any hills. You could shift all the distances by a scaling factor α. These are continuous transformations. For all of these you can see if the transformation effects the evaluation of the criteria.

From this example you see that you need both a set of transformations and a criteria.

In physical systems, we can either change the events in the transformation process or change the measuring system that is used to identify the events. The former case is called the active view of transformations and the latter is the passive view. Obviously, they are equivalent descriptions of the effects of the transformations and which is being used is chosen by the context of the problem.

7.3 Examples of Symmetry in physics

In physics we are interested in what happens to things in space time, i. e. events. These are labeled by (x,t). An event is a point in a space time diagram. A connected set of events is a trajectory. This is the path that a particle follows as it moves. This is often called a particles world line.

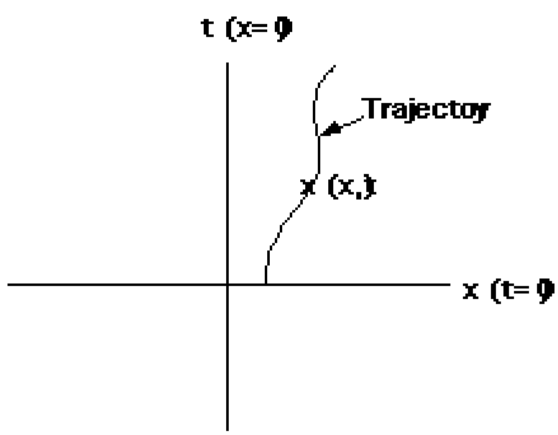

Figure 7.8: **Action trajectory Trajectory 2**

7.3.1 Physics transformations:

Space Reflection:

This is the transformation that corresponds to the bilateral transformation that we discussed earlier. We reflect all the events through the line $x = 0$ better known as the t axis.

$$x \rightarrow x' = -x \tag{7.1}$$

I am showing this transformation in the active view.

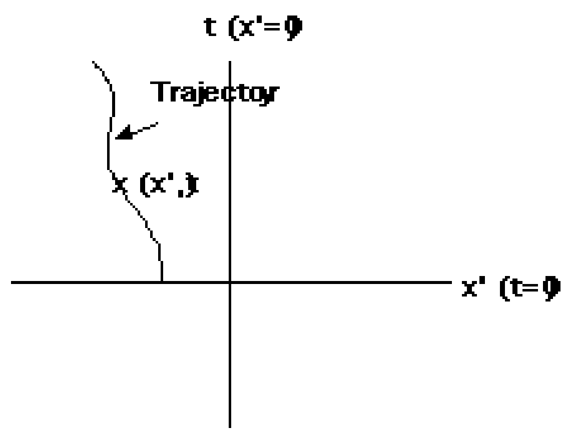

Figure 7.9: **Space Reflection** Space Reflection

Space Translation:

Shift the origin of the coordinate system.

$$x \rightarrow x' = x + a \tag{7.2}$$

Time Translation:

Shift the start of the time.

$$t \rightarrow t' = t + a \tag{7.3}$$

. To be a symmetry we will require that the physics before and after the shift is the same. I have not carefully defined what I mean by "the same." I will do so shortly.

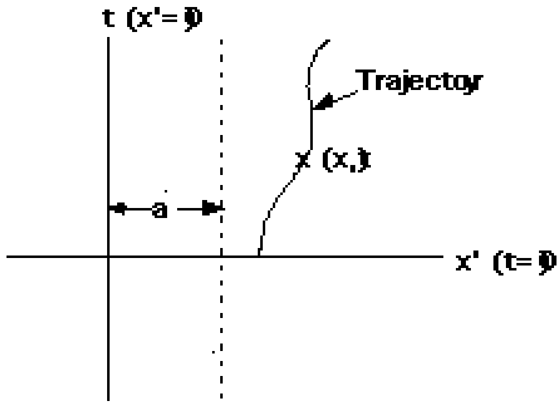

Figure 7.10: **Space Translation** Space Translation

Newton's Action at a Distance Law of Gravitation

The law of force that describes the gravitational influence of one body, say body 2, on another body, say body 1, is

$$\vec{F}_{1,2} = G\frac{m_1 m_2}{|\vec{r}_2 - \vec{r}_1|^3} \times (\vec{r}_2 - \vec{r}_1) \tag{7.4}$$

Similarly, the gravitational force of body 1 on body 2 can be found by interchanging the labels of particles 1 and 2.

$$\vec{F}_{2,1} = G\frac{m_2 m_1}{|\vec{r}_1 - \vec{r}_2|^3} \times (\vec{r}_1 - \vec{r}_2) \tag{7.5}$$

Thus if you are operating at the level of the forces you have that if you interchange particles 1 and 2, i. e. change the labels 1 and 2, $1 \leftrightarrow 2$ and get $\vec{F}_{1,2} \to -\vec{F}_{2,1}$ This is a discrete transformation. If for some reason you are interested in the forces, this is not a symmetry. It is actually a manifestation of the Law of Action Reaction. In other words, we construct the Law of Gravitation so that it obeys the Law of Action Reaction. On the other hand, if you look at the entire set of equationswithout the forces, there is no change.

$$m_1\vec{a}_1 = G\frac{m_1 m_2}{|\vec{r}_2 - \vec{r}_1|^3} \times (\vec{r}_2 - \vec{r}_1) \tag{7.6}$$

$$m_2\vec{a}_2 = G\frac{m_2 m_1}{|\vec{r}_1 - \vec{r}_2|^3} \times (\vec{r}_1 - \vec{r}_2) \tag{7.7}$$

Some symmetries of this law:

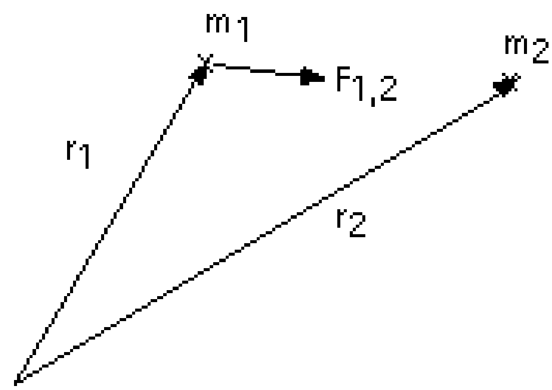

Figure 7.11: **Gravitational Force between Two Bodies**

This is then a symmetry. When you put a shift to all the positions by some amount, $\vec{a}$, nothing changes, i. e. $\vec{r}_i \to \vec{r}_i + \vec{a}$. This is a continuous symmetry. When you replace all the positions with the reverse position, $\vec{r}_i \to -\vec{r}_i$ again nothing changes. Remember $\vec{a}_i \to -\vec{a}_i$. This is a discrete symmetry. If you change all the distances in the problem by a scale $\vec{r}_i \to \vec{r'}_i = \lambda \vec{r}_i$, then this is not a symmetry. But, if you also change the time scale by $t \to t' = \lambda^{\frac{3}{2}} t$, then you have a symmetry. This is a continuous symmetry. Note that the identity transformation is $\lambda = 1$.

7.4 Symmetry and Action

7.4.1 Introduction

You can have the situation that you make the change and the action does not change at all. Said more carefully, you have transformed end points and transformed paths and you get the same value for the action. Consider the free particle and translations in space.

$$\begin{aligned} x' &= x + a \\ t' &= t \end{aligned} \tag{7.8}$$

This implies that $v' = v$. Thus

$$\begin{aligned} S'(x'_f, t'_f, x'_0, t'_0; path') &= \sum_{path',(x'_0,t'_0)}^{(x'_f,t'_f)} (m\frac{v'^2}{2})\Delta t \\ &= \sum_{path,(x_0,t_0)}^{(x_f,t_f)} (m\frac{v^2}{2})\Delta t \\ &= S(x_f, t_f, x_0, t_0; path) \end{aligned} \quad (7.9)$$

If action is the basis of all physics, then we have a natural definition of a symmetry of a physical system. A physical system has a symmetry if there is a way to modify the system and yet there is no significant change in the action. It is important to be careful about the meaning of significant in this sentence. For most purposes the value of the action is not important. The action primary role is to select a path from the infinity of possibilities. In this sense, we can as a first step assert that the system is symmetric if the system before and after the change still selects the same path as the natural path. You again have to be careful because the same path is actually the same path as seenin the modified system. An example might help clarify this.

Harmonic Oscillator and Symmetry

The harmonic oscillator is one of the most important physical systems. We will discuss the physics of this system in greater detail in a later section, Section 8.2 on page 205, but for now will use it as another example in which to examine the role of symmetry in a physical system. For now just think of it as a physical system that goes back and forth.

The Lagrangian for the harmonic oscillator is

$$L(v, x) = KE - PE = m\frac{v^2}{2} - k\frac{x^2}{2} \quad (7.10)$$

where k is the spring constant and m is the mass and both are given constants and have the dimension $k \stackrel{\text{dim}}{=} \frac{\text{Mass}}{\text{Time}^2}$ and, of course, m is a mass. Note that, if these are the only two dimensional constants that are available, then you cannot make a length but you can make a time. If you rescale the distances by an amount λ, as follows:

$$\begin{aligned} x &\rightarrow x' = \lambda x \\ t &\rightarrow t' = t \end{aligned} \quad (7.11)$$

which implies that

$$v \to v' = \frac{\Delta x'}{\Delta t'} = \lambda \frac{\Delta x}{\Delta t} = \lambda v \tag{7.12}$$

The Lagrangian for the new system is

$$L'(v', x') = KE' - PE' = m\frac{v'^2}{2} - k\frac{x'^2}{2} = m\lambda^2 (\frac{v^2}{2} - k\frac{x^2}{2}) = \lambda^2 L(v, x) \tag{7.13}$$

So that

$$\begin{aligned} S'_{Path'}(x'_0, t'_0; x'_f, t'_f) &= \sum_{path', (x'_0, t'_0)}^{(x'_f, t'_f)} (m\frac{v'^2}{2} - k\frac{x'^2}{2})\Delta t' \\ &= \lambda^2 \sum_{path, (x_0, t_0)}^{(x_f, t_f)} (m\frac{v^2}{2} - k\frac{x^2}{2})\Delta t \\ &= \lambda^2 S_{Path}(x_0, t_0; x_f, t_f) \end{aligned} \tag{7.14}$$

where Path' is the Path that is at the rescaled distances

$$x'(t') = \lambda x(t) \tag{7.15}$$

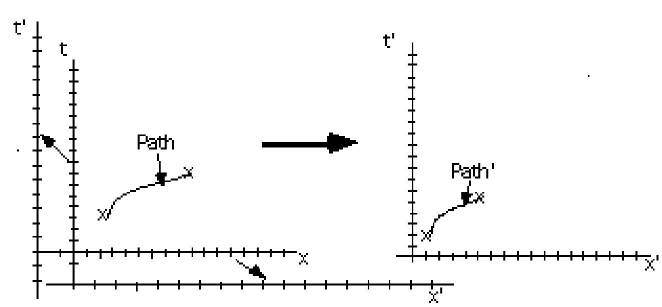

Figure 7.12: **Rescale Oscillator** Rescale Oscillator

Path	Action	Path'	Action'
1	S_1	1'	$S'_{1'} = \lambda^2 S_1$
2	S_2	2'	$S'_{2'} = \lambda^2 S_2$
.	.	.	.
.	.	.	.
.	.	.	.
natural	S_{least}	natural'	$S'_{least'} = \lambda^2 S_{least}$
.	.	.	.
.	.	.	.
.	.	.	.

You get the same path even though the calculations are all different.

7.4.2 Galilean Invariance

In order to show that the straight line was the solution to the free particle action problem I assumed that the action procedure was Galilean invariant and went to a special frame. The question is "is it." The action is

$$S(x_f, t_f, x_0, t_0; traj.) = \sum_{traj., x_0, t_0}^{x_f, t_f} \left(m \frac{v^2}{2} \right) \Delta t \tag{7.16}$$

What happens when you make the Galilean transformation?

$$\begin{aligned} x' &= x - at \\ t' &= t \end{aligned} \tag{7.17}$$

Where a is a parameter that labels the transformations and has the dimensions of a velocity – it is actually interpreted as a velocity. With this transformation all the velocities shift, $v' = v - a$ because $v \equiv \frac{x(t+\Delta t)-x(t)}{\Delta t}$ and Inserting this into

$$\begin{aligned} v' &\equiv \frac{x'(t'+\Delta t') - x'(t')}{\Delta t'} = \frac{x(t+\Delta t) - a\{t+\Delta t\} - x(t) + a\{t\}}{\Delta t} \\ &= \frac{x(t+\Delta t) - x(t)}{\Delta t} - \frac{a\Delta t}{\Delta t} \\ &= v - a. \end{aligned} \tag{7.18}$$

Plugging this into the action,

$$S'(x'_f, t'_f, x'_0, t'_0; path') = \sum_{traj.', x'_0, t'_0}^{x'_f, t'_f} \left(m \frac{v'^2}{2} \right) \Delta t'$$

$$
\begin{aligned}
&= \sum_{traj.,x_0,t_0}^{x_f,t_f} \Big(m\frac{(v-a)^2}{2}\Big)\Delta t \\
&= \sum_{traj.,x_0,t_0}^{x_f,t_f} \Big(m\frac{v^2}{2}\Big)\Delta t - \sum_{traj.,x_0,t_0}^{x_f,t_f} (mva)\Delta t + \sum_{traj.,x_0,t_0}^{x_f,t_f} \Big(m\frac{a^2}{2}\Big)\Delta t \\
&= S(x_f,t_f,x_0,t_0;traj.) - ma \sum_{traj.,x_0,t_0}^{x_f,t_f} v\Delta t + \Big(m\frac{a^2}{2}\Big) \sum_{traj.,x_0,t_0}^{x_f,t_f} \Delta t \\
&= S(x_f,t_f,x_0,t_0;traj.) - ma(x_f - x_0) + \Big(m\frac{a^2}{2}\Big)(t_f - t_0)
\end{aligned}
$$

The action of any trajectory is different from the action for the transformed trajectory. Since the action changes the best that could happen is an invariance. The last two terms are independent of trajectory. Therefore the trajectory selection process that selects the least action trajectory in S will select the transformed trajectory in S'. The action changes under the transformation but in an unimportant way. This is not a symmetry and there is no associated conserved quantity. When we implement this for special relativity it will become a symmetry.

7.4.3 More on Symmetry and Action

The easiest way to guarantee that the action is symmetric under a set of transformations is to construct it only from the form invariants for that set of transformations. In fact, it is a necessary and sufficient condition that the action is symmetric that it be composed of only form invariants for that set of transformations.

As an example consider the action for a satellite of mass m in orbit around the earth. Locating the earth at the origin, the action is

$$
S(\vec{x}_0, t_0, \vec{x}_f, t_f; path) = \sum_{Path,\vec{x}_0,t_0}^{\vec{x}_f,t_f} (m\frac{\vec{v}^2}{2} + Gm\frac{M_{earth}}{r})\Delta t \qquad (7.19)
$$

This action is composed of $\vec{v}^2$ which is a form invariant for rotations about the origin. r is the distance from the origin and it is also a form invariant for rotations. Obviously Δt is a form invariant for rotations. Thus this action has a symmetry that is the set of transformations that are the rotations about the origin.

7.4.4 Noether's Theorem

For every transformation family whose members are labeled by an element of an $\mathbf{R}^1$ *and whose members are connected smoothly to the identity transformation that is a symmetry there exists a conserved quantity during the naturally occurring trajectory. Noether's Theorem also tells us how to construct the conserved quantity.*
Agreed. This is a most opaque statement and will require some clarification. When I tell you what the question is and thus when a change is important, I can tell you how to construct the conserved quantity.

Space translation Symmetry

The conserved quantity that is associated with situations with space translation symmetry is called linear momentum. In certain cases it is $\vec{p} = m\vec{v}$ but not all the time. I will tell you when those cases are.

Rotation translation symmetry

The conserved quantity that is associated with situations with space rotation symmetry is called angular momentum. Rotations are a vector quantity. Again in certain cases it is $\vec{L} = m\vec{r} \times \vec{v}$.

Time translation Symmetry

The conserved quantity that is associated with situations with time translation symmetry is called energy. This is actually the case all the time but the form of the energy may change.

Galilean Invariance*

This is almost a symmetry classically and becomes a full blown symmetry in the modern language. First, let's discuss what the transformation is.

There is no experiment that can be performed that can measure the velocity of a uniformly moving observer. We can detect the presence of accelerations and measure the relative velocity between two bodies but we cannot measure velocities absolutely.

Another way to say the same thing is that, if you are not accelerating, you are always at rest in your own rest frame. It is a requirement for consistency that then leads to the special category of observers[1] that have no acceleration. Their observations always must agree.

In the language of transformations, all the laws of physics must be invariant under transformations of the form

$$\begin{aligned} \vec{x} &\rightarrow \vec{x}' = \vec{x} + \vec{R} + \vec{v}t \\ t &\rightarrow t' = t \end{aligned} \tag{7.20}$$

where $\vec{R}$ and $\vec{v}$ are constants that are the parameters that label these continuous transformations. Unfortunately the parameter $\vec{v}$ is often called a velocity because dimensionally it is a dim LT which has another identification as a velocity. These parameters can be interpreted in terms of two coordinate systems this can be interpreted as the measurements of two relatively displaced and relatively moving coordinate systems.

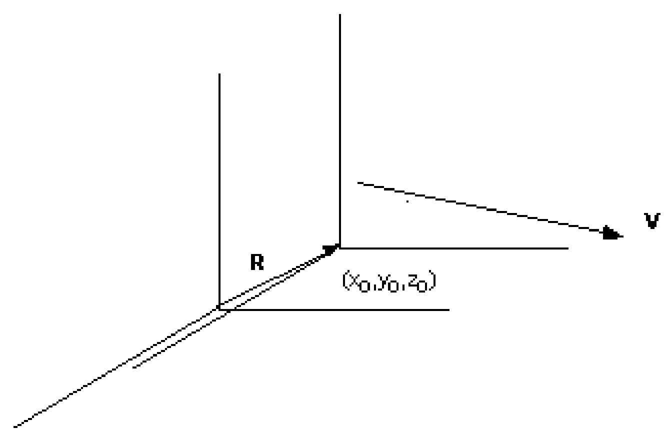

Figure 7.13: **Galilean Invariance** Galilean Invariance

Although this is a continuous symmetry that is connected with the identity, $\vec{R} = \vec{v} = 0$, it is not a symmetry classically. I will explain this later. Since this is not a symmetry, there is no conserved quantity that is the result of Galilean invariance in classical physics.

You should apply this transformation to the gravitational force above and see that neither the forces nor the equations change. If you use these as your criteria for a symmetry, this would be a symmetry. It is not so we see that we need a better criteria.

[1]The idea of observer is a subtle and important one. The observer is a general title for a measuring system. Sorry for the anthropomorphism. The other important point that all observers that are at rest have the same results for their experiments; they are good physicists.

Please read the Feynman's action lecture. I do not expect that all of you will follow this material in detail. Just get the drift.

Consider a change in the system that also changes the description of initial and final events. This is what will generally happen. Here, when you do the transformations, you will get in addition to the usual terms of the integral of the Lagrangian but also terms from the end points. Our modified form of Feynman's equation

$$\begin{aligned}\delta S &= \left(\left(\frac{\delta L}{\delta v}\,|_{x_{nat}(t)}\right)\delta x\right)_{t_f} - \left(\left(\frac{\delta L}{\delta v}\,|_{x_{nat}(t)}\right)\delta x\right)_{t_0} \\ &\quad + \int_{t_0}^{t_f}\left(\frac{d}{dt}\left(\frac{\delta L}{\delta v}\right) - \frac{\delta L}{\delta x}\right)\delta x dt \end{aligned} \tag{7.21}$$

To get the action to be stationary now we will require that as before the integrand vanish

$$\frac{d}{dt}\left(\frac{\delta L}{\delta v}\right) - \frac{\delta L}{\delta x} = 0 \tag{7.22}$$

but also that the terms from the end points vanish. This part simply selects the natural path. To understand the end points consider an example, the simple translation. In this case δx is simply a number that is added to all points in the path.

$$\delta x(t_f) = \delta x(t_0) = a \tag{7.23}$$

or

$$\left(\left(\frac{\delta L}{\delta v}\,|_{x_{nat}(t)}\right)\delta x\right)_{t_f} - \left(\left(\frac{\delta L}{\delta v}\,|_{x_{nat}(t)}\right)\delta x\right)_{t_0} = \left(\left(\frac{\delta L}{\delta v}\,|_{x_{nat}(t)}\right)_{t_f} - \left(\frac{\delta L}{\delta v}\,|_{x_{nat}(t)}\right)_{t_f}\right) a \tag{7.24}$$

Setting this to zero, yields

$$\left(\frac{\delta L}{\delta v}\,|_{x_{nat}(t)}\right)_{t_f} = \left(\frac{\delta L}{\delta v}\,|_{x_{nat}(t)}\right)_{t_f} \tag{7.25}$$

But $\frac{\delta L}{\delta v}\,|_{x_{nat}(t)}$ is what you would **define** as the momentum. It is the momentum when you use the usual Lagrangian. Thus thisis nothing more than the statement that momentum is conserved.

$$p(t_f) = p(t_0) \tag{7.26}$$

This is a special case of a general theorem called Noether's Theorem. Given any transformation that can be connected with the identity transformation, no change, by a continuous parameter. There will always be a conserved quantity. In the above example the transformation is translation. In the limit $a \to 0$ you have no translation and thus no change and the identity transformation. In this case, the conserved quantity is the linear momentum.

Another way of looking at this result is that, once you have selected the natural path and if you include the end point variations, the action is a function of the end points only. If the symmetry transformation changes the end points you have

$$\delta S_{Nat}(x_0, t_0; x_f, t_f) = \frac{\delta S_{Nat}}{\delta x_0}\delta x_0 + \frac{\delta S_{Nat}}{\delta x_f}\delta x_f + \frac{\delta S_{Nat}}{\delta t_0}\delta t_0 + \frac{\delta S_{Nat}}{\delta t_f}\delta t_f \quad (7.27)$$

In the case of translations,

$$\delta x(t_f) = \delta x(t_0) = a \quad (7.28)$$

and all the δt_i are zero.

Thus we get

$$\frac{\delta S}{\delta x_f} = -\frac{\delta S}{\delta x_0} = p = \text{constant} \quad (7.29)$$

An Example

For the free particle,

$$S_{natural} = m\frac{(x_f - x_0)^2}{2(t_f - t_0)} \quad (7.30)$$

$$p = \frac{\delta S}{\delta x_f} = m\frac{(x_f - x_0)}{(t_f - t_0)} = mv \quad (7.31)$$

since v is a constant.

We noted above that the satellite in orbit is a case that is invariant under rotations about the origin. This set of transformations is a continuous set and thus there is a conserved quantity. In this case we call it the angular momentum. The construction of this conserved quantity involves cumbersome notation because it only makes sense in a system with at least two spatial dimensions and thus involves vector notation. In addition, it is computationally difficult to find an expression for the natural path. But note that the free particle Lagrangian is also composed only of form invariants for rotations about the origin. Thus this set of transformations is also a symmetry for this case. The analysis is still cumbersome because of the vector notation. I am aware that you will not be able to reproduce this analysis. All that I ask is that you follow it.

We will work in two spatial dimensions. For this case the action is

$$S(\vec{x}_0, \vec{t}_0; \vec{x}_f, \vec{t}_f) = \sum_{\text{NaturalPath}, \vec{x}_0, \vec{t}_0}^{\vec{x}_f, \vec{t}_f} m \frac{\vec{v}^2}{2} \Delta t \tag{7.32}$$

and as we see is composed of only form invariants not only of translations in space and time but also for rotations. The quantity $\vec{v}^2$ is invariant under rotations.

For the natural path the action is

$$S_{natural} = m \frac{(\vec{x}_f - \vec{x}_0)^2}{2(t_f - t_0)} \tag{7.33}$$

and the change in the action caused by the end point changes are

$$\delta S_{Nat}(\vec{x}_0, t_0; \vec{x}_f, t_f) = \frac{\delta S_{Nat}}{\delta \vec{x}_0} \cdot \delta \vec{x}_0 + \frac{\delta S_{Nat}}{\delta \vec{x}_f} \cdot \delta \vec{x}_f + \frac{\delta S_{Nat}}{\delta t_0} \delta t_0 + \frac{\delta S_{Nat}}{\delta t_f} \delta t_f \tag{7.34}$$

For rotations, δt_0 and δt_f are zero. The $\delta \vec{x}_0$ and $\delta \vec{x}_f$ are the displacements of the end points that result from the rotation. For a rotation through an angle θ, they are

$$\delta \vec{x}_0 \tag{7.35}$$

From the rule above we need the change in the S_{Nat} along this direction.

As in the translation example we see that the change in S with changes in position is the regular momentum. Thus the thing that multiplies $\delta\theta$ in the change in action is the momentum along this direction times the distance. This is what we always called the angular momentum.

Thus we get the rather complicated object

$$L_{axis} = \frac{\delta S_{Nat}}{\delta \vec{x}_0} \cdot r_0 \left(\theta\right)_0 \tag{7.36}$$

The lesson of all this is that the symmetry implies that there is a conserved quantity. These are the things that we call momenta or energy etc. The form that they take depends on the nature of the Lagrangian.

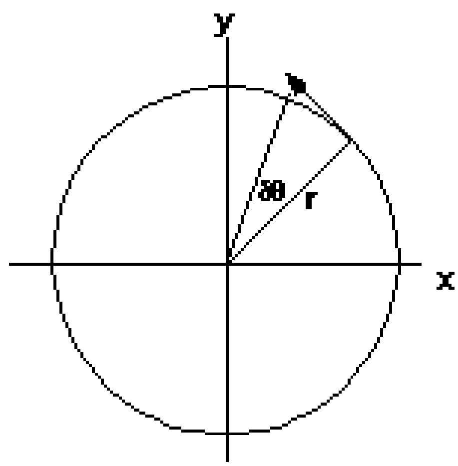

Figure 7.14: **Rotation** Rotation.